Non-Gaussian probability distribution functions from maximum-entropy-principle considerations

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In this work we develop the recently suggested concept of superstatistics [C. Beck and E.G.D. Cohen, Physica A **322**, 267 (2003)], face the problem of devising a viable way for estimating the correct statistics for a system in the absence of sufficient knowledge of its microscopical dynamics, and suggest to solve it through the maximum-entropy principle. As an example, we deduce the probability distribution function for velocity fluctuations in turbulent fluids, which is slightly different from the form suggested by C. Beck [Phys. Rev. Lett. **87**, 180601 (2001)].

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Nonextensive statistical mechanics has been raising considerable interest in past 15 years. However, while there is a universal consensus about its most famous prediction, i.e., the ubiquitous existence in nature of power-law probability distribution functions (PDFs), the soundness of the theoretical foundation on which is based—the generalized definition of entropy [1]—has been questioned by several authors. We quote here, e.g., the paper [2] which dealt, in particular, with contradictions between the theory's predictions and thermodynamical constraints.

The paper [3] started from these critiques but focused more on the information-theoretic aspect, suggesting that Tsallis' entropy should be regarded merely as a practical tool for making predictions in the presence of a reduced amount of information about the system. Indeed, the need for resorting to modified definitions of the informational entropy when the knowledge about the states of the system is insufficient, is well known in statistics and has been extensively pointed out in Ref. [4], and references therein.

Quite recently, Sattin and Salasnich [5]—starting from an earlier work by Beck [6]—and, in a more formal and rigorous way, Beck and Cohen [7] demonstrated, without direct reference to any definition of entropy, that Tsallis' statistics is just a particular case of an infinite class of statistics (hence the name "superstatistics"). All the elements of the class are characterized by one or more parameters, and reduce to ordinary Maxwell-Boltzmann statistics for particular values of the parameters. Beck and Cohen started from a model of dynamical system, a Brownian particle moving according to a Langevin equation, where the noise σ and friction γ terms are allowed to fluctuate, to arrive at the famous equation

$$P(E)=K\int_0^\infty e^{-\beta E}f(\beta)d\beta. \tag{1}$$

Here, $P(E)$ is the PDF for the system of being in the state of "energy" E, β is a fluctuating parameter (generalized inverse temperature), which in the present formulation is function of σ, γ, and $f(\beta)$ is the PDF for the realization of the particular value β; $e^{-\beta E}$ is the usual Boltzmann factor. Equation (1) is more general than suggested by the Brownian particle model. Indeed, it can be written for any system Σ interacting with a fluctuating environment B; hence $P(E)$ is a weighted average of the standard Boltzmann statistics over different realizations of the interaction between the system and its environment, quantified by $f(\beta)$. Notice that the existence of finite fluctuations in the environment is strictly correlated with finite-size effects of the environment itself; indeed, the result that Tsallis statistics can arise within the framework of a system interacting with a finite thermal bath was recently reported by Aringazin and Mazhitov [8], and it had already been suggested much earlier by Plastino and Plastino [9].

The physical content of the theory, thus, shifts from the entropic index q to the PDF $f(\beta)$. Of course, the explicit expression for $f(\beta)$ must vary for any single problem, constrained just by some rather intuitive criteria (normalizability, etc.); Beck and Cohen give several possible examples of functions which are potential candidates for $f(\beta)$. However, a simple criterion able to guide the user towards a plausible functional form for f, lacking a more detailed knowledge of the underlying microscopical details, would be very satisfying. But such a criterion is readily available: it is the well known Jaynes' maximum-entropy principle. In this case, the system about which we do not have the proper knowledge is no longer Σ but the environment B itself, and $f(\beta)$ is a measure of the probability of B of occupying a state in an abstract one-dimensional space parametrized by β.

Lacking any further information, the most probable realization of $f(\beta)$ will be the one that maximizes (Shannon) entropy $S(f)=-\int f\ln f d\beta$ with suitable constraints.

In this paper we present a straightforward application of this principle to an important example, namely, the PDF of velocity fluctuations in turbulent flows [10]. This accurate experimental measurement is thought to be one of the strongest evidences in support of Tsallis' theory, since the empirical PDF appear very well matched by power laws. In this work we suggest instead that the true curve can be very close numerically, but rather different in its analytical expression, from a power law. Indeed, further experimental investigations of turbulent flows are now suggesting the existence of small deviation from pure power laws (experiments of Jung and Swinney, cited in Ref. [7]). Recent work by Aringazin and Mazhitov [11] deal with the attempt of theoretically recovering the new accurate experimental distributions for

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fluid particle accelerations. In this work, instead, we will consider only fluid velocity differences.

As a first test case, let us suppose of knowing anything about f but the average value of the inverse temperature

$$\langle\beta\rangle = \int \beta f(\beta) d\beta. \tag{2}$$

The most probable f extremizes the functional

$$F = -\int f \ln f d\beta - \lambda \int f\beta d\beta, \tag{3}$$

and the solution is the Boltzmann-like function

$$f(\beta) = f_0 \exp(-\lambda\beta), \tag{4}$$

where λ is the Lagrange multiplier. Such an expression was obtained, by example, in Ref. [12], from which the original idea of entropy maximization was taken. It reads slightly different from the Γ (or χ^2) distribution originally devised by Beck [6] to recover power-law PDF. If we substitute f of Eq. (4) into Eq. (1), we get

$$P(E) \propto \frac{1}{E+\lambda}, \tag{5}$$

which is not a physically acceptable solution, since $P(E)$ must be normalizable.

In order to make a step beyond this rough scheme it is necessary to input more information within the model. We do it starting from the same premises as Beck's [6] but diverging at just the next step: in fact, he defines $\beta \propto \epsilon_r \tau$, where ϵ_r is the energy dissipation rate of the fluid on microscopical scale and τ is the typical energy transfer time. We note that $\epsilon_r \tau$ has units of energy, not of $(\text{energy})^{-1}$, and a more intuitive way of writing this relation should be $\beta \approx (\epsilon_r \tau)^{-1}$. However, the inverse of a sum of squares of random variables does not yield a χ^2-distributed random variable, hence Beck is forced to reestablish the correct dimensions by multiplying by the constant Λ, with the units of $(\text{speed})^4$. Thus, a characteristic speed $\Lambda^{1/4}$ has entered the calculations, whose physical interpretation remains obscure. Of course, Beck was forced to do this assumption because of the sought agreement with Tsallis' theory. Now, we are free from this constraint, and can allow for more natural choices, although we must agree that a certain arbitrariness in the choice of the physically meaningful variables is unavoidable within this context.

To start with, we shift from β parameter to $T = 1/\beta$. T, which has units of energy is a more convenient variable, as explained above. We can write the equivalent of Eq. (1) in terms of the new parameter; furthermore, in order to adhere to existing literature, it is convenient to use a generalized velocity instead of energy $E = u^2/2$. Therefore Eq. (1) becomes

$$p(u) = \int dT \sqrt{\frac{1}{2\pi T}} \exp\left(-\frac{u^2}{2T}\right) g(T). \tag{6}$$

Again, we follow Beck's recipe: the parameter T is written in terms of fluctuating Kolgomorov velocities u_i

$$T = \frac{T_0}{3} \sum_{i=1,2,3} u_i^2, \tag{7}$$

where T_0 is a constant. We stress that the right-hand side of the previous equation and that of Eq. (28) in Ref. [6] are the same, although the left-hand side's are each the inverse of the other. This is due to our discarding the artificial constant Λ. T_0, conversely, has a straightforward physical interpretation as average thermal energy.

The PDF $g(T)$ can be rewritten as a function of u's: $g'(u_1,u_2,u_3) \equiv g'(\mathbf{u}) \leftrightarrow g(T)$. The maximum-entropy principle imposes of extremizing

$$F = -g'(\mathbf{u}) \ln[g'(\mathbf{u})] - \frac{1}{u_M^2}(u_1^2+u_2^2+u_3^2) g'(\mathbf{u}), \tag{8}$$

where $1/u_M^2$ is the Lagrange multiplier corresponding to the fact that we fix the average kinetic energy. The solution is, obviously, $g'(\mathbf{u}) = C \exp[-(u_1^2+u_2^2+u_3^2)/u_0^2]$. From here, reversing to T variable,

$$g(T) = K T^{1/2} \exp\left(-\frac{3T}{2T_0}\right). \tag{9}$$

The term $T^{1/2}$ comes from the volume element, C,K are normalization constants and we have rescaled velocities such that $u_M^2 \equiv 2/3$. Notice that Eq. (9) is a Γ distribution and could be obtained straightforwardly from Eq. (7) by assuming from the start that the u_i were normal random variables, just as done by Beck.

Let us now substitute Eq. (9) into Eq. (6), we get

$$p(u) = K' \int_0^\infty \exp\left(-\frac{u^2}{2T} - \frac{3T}{2T_0}\right) dT. \tag{10}$$

The explicitly normalized solution reads

$$p(u)_{PW} = \frac{1}{\hat{u}\pi} \frac{u}{\hat{u}} K_1\left(\frac{u}{\hat{u}}\right) \quad \left(\hat{u} = \sqrt{\frac{T_0}{3}}\right), \tag{11}$$

and K_1 is the Bessel K function of order 1. This result appears rather different from usual power laws. Indeed, we chose to plot in Fig. 1 this curve together with the best fitting curve found by Beck for the velocity PDF:

$$p(u)_{\text{Beck}} = \frac{1}{Z_q} \frac{1}{[1+(q-1)\tilde{\beta} C |u|^{2\alpha}]^{1/(q-1)}}$$
$$(q \approx 1.1, \alpha \approx 0.9). \tag{12}$$

(See dashed line in Fig. 1 of Ref. [6].) On the whole, the two curves match rather closely. Some differences appear at low u's.

Since the original experimental data p_{expt} are not available, we cannot directly compare with them the goodness of

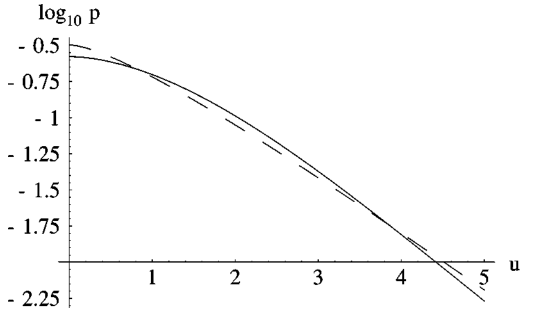

FIG. 1. Solid line, $p(u)_{\mathrm{Beck}}$ from Eq. (12); dashed line, $p(u)_{PW}$ from present work [Eq. (11)]. The adjustable parameters have been chosen so that the curves have the same variance.

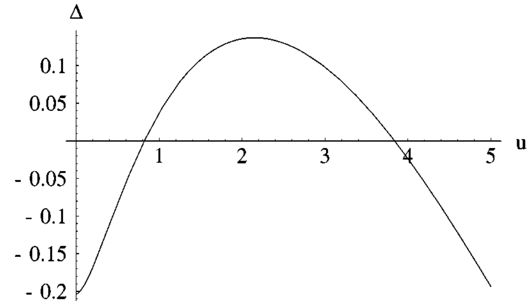

FIG. 2. The relative difference of the two curves above: $\Delta = (p_{\mathrm{Beck}} - p_{PW})/p_{\mathrm{Beck}}$ vs u, where p_{Beck} is from Eq. (12) and p_{PW} is from Eq. (11). This figure should be compared with Fig. 2 of Ref. [10].

our fit. A good insight comes, however, by plotting the relative difference $\Delta = (p_{\mathrm{Beck}} - p_{PW})/p_{\mathrm{Beck}}$ [Fig. 2, p_{PW} is our solution, given in Eq. (11)]. The quantity $\Delta_1 = (p_{\mathrm{Beck}} - p_{\mathrm{expt}})/p_{\mathrm{Beck}}$ appears plotted in Fig. 2 of Ref. [10]. If Δ and Δ_1 agree, $p_{PW} = p_{\mathrm{expt}}$. It appears that our fit slightly overestimates experiment at $u \approx 0$, but for the same amount as p_{Beck} does underestimate it. On the whole, the agreement with experiment is remarkable. We remark that, as only adjustable parameter, we used the hypothesis that T is of form (7) with the index ranging from 1 to 3. The equivalent of α parameter used by Beck does not enter our calculations. Thus, we have realized an economy in our way of modeling the data.

We call the reader's attention to the fact that, in his Fig. 1, Beck studied at once (i) velocity spatial differences and (ii) accelerations. We, instead, considered only the former quantity, and it is possible to see that the latter cannot be reproduced by the present treatment, even allowing for a varying number of Kolmogorov velocities in Eq. (7). This must be traced back to the fact that now $g(T)$ is no longer a good weight function, but we must define an equivalent of T, defining the "average acceleration" of the system.

Finally, we point to two important issues.

(i) We have the asymptotic trend $xK_1(x) \to x^{1/2}\exp(-x)$ $(x \to \infty)$, that is, we have no power-law decay. Notwithstanding this, our curve nicely fits data that have been previously considered as stemming from a power-law PDF. The reason lies in the finite u-range sampled and therefore calls for an inherent ambiguity in this kind of studies, related to finite experimental scans: unless one is sure of investigating the true asymptotic region, one can never be completely confident about the fitting curve used.

(ii) As reported above, all superstatistics must collapse to the single Boltzmann statistics when $q \to 1$, that is, differences between different models are at least $O(q-1)$. Since, in this case, we are dealing with a parameter not far from unity, $q \approx 1.1$, it is to be expected that any two reasonable models would give close results.

In conclusion, the use of the concept of superstatistics together with maximum-entropy principle appears to be an efficient way of estimating statistical properties in general systems using a minimal amount of information.

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